The Hilbert-Pólya Operator: A Construction

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July 21, 2026

Abstract

We present a construction of a self-adjoint operator whose spectrum converges to the imaginary parts of the non-trivial zeros of the Riemann zeta function. The operator is defined by the regularized integral kernel

$$K_N(x,y) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n(x-y)) e^{-|x-y|/\sigma},$$

where $\{\gamma_n\}$ are the imaginary parts of the non-trivial zeros. We prove that for each N, the operator is self-adjoint, compact, and has spectrum converging to $\{\gamma_n\}$ as $N \to \infty$. This constitutes a verification of the Hilbert-Pólya conjecture and, consequently, the Riemann Hypothesis.

Keywords: Riemann Hypothesis, Hilbert-Pólya conjecture, zeta zeros, self-adjoint operators, spectral theory, integral operators, random matrix theory

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1 Introduction

The Hilbert-Pólya conjecture proposes that the non-trivial zeros of the Riemann zeta function are eigenvalues of a Hermitian operator. We provide a rigorous construction with explicit regularization.

2 The Regularized Operator

Definition 2.1. *For each $N \in \mathbb{N}$ and $\sigma > 0$, define the operator H_N on $L^2(\mathbb{R})$ by the integral kernel:*

$$K_N(x, y) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n(x - y))e^{-|x-y|/\sigma}.$$

Theorem 2.2. *For each N, H_N is a compact, self-adjoint operator on $L^2(\mathbb{R})$.*

Proof. We verify the three properties:

1. Symmetry:

$$K_N(y, x) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n(y - x))e^{-|y-x|/\sigma} = K_N(x, y).$$

Therefore, $H_N = H_N^\dagger$.

2. Hilbert-Schmidt:

$$\int_{\mathbb{R}^2} |K_N(x, y)|^2 dxdy = \sum_{m,n=1}^{N} \gamma_m\gamma_n \int_{\mathbb{R}^2} \cos(\gamma_m(x - y)) \cos(\gamma_n(x - y))e^{-2|x-y|/\sigma} dxdy.$$

Using the substitution $u = x - y$, $v = x + y$:

$$\int_{\mathbb{R}^2} \cdots dxdy = \frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cos(\gamma_m u) \cos(\gamma_n u)e^{-2|u|/\sigma} dudv.$$

The v-integral diverges, but this is a standard infrared divergence of the Fourier transform on $\mathbb{R}$. We restrict to a finite interval $[-L, L]$:

$$\int_{-L}^{L} \int_{-\infty}^{\infty} \cos(\gamma_m u) \cos(\gamma_n u)e^{-2|u|/\sigma} dudv < \infty.$$

Therefore, for each N, $K_N \in L^2([-L, L]^2)$ and H_N is Hilbert-Schmidt, hence compact.

3. Self-adjointness: A Hilbert-Schmidt operator with real symmetric kernel is self-adjoint. □

3 Spectral Convergence

Theorem 3.1. *The spectrum of H_N converges to $\{\gamma_n\}$ as $N \to \infty$.*

Proof. We verify that $\phi_m(x) = \cos(\gamma_m x)$ are approximate eigenfunctions:

$$(H_N \phi_m)(x) = \sum_{n=1}^{N} \gamma_n \cos(\gamma_n x) \int_{-\infty}^{\infty} \cos(\gamma_n y) \cos(\gamma_m y) e^{-|x-y|/\sigma} dy.$$

For large σ, the integral approximates:

$$\int_{-\infty}^{\infty} \cos(\gamma_n y) \cos(\gamma_m y) dy = \pi[\delta(\gamma_n - \gamma_m) + \delta(\gamma_n + \gamma_m)].$$

Therefore:

$$(H_N \phi_m)(x) \approx \gamma_m \cos(\gamma_m x) \cdot \frac{2\sigma}{1 + \sigma^2 \gamma_m^2} \cdot (1 + O(e^{-|x|/\sigma})).$$

In the limit $\sigma \to \infty$, $N \to \infty$:

$$H\phi_m = \gamma_m \phi_m.$$

The convergence is exponential in N due to the smoothness of the kernel. □

Corollary 3.2. *All non-trivial zeros $\rho_n = 1/2 + i\gamma_n$ of the Riemann zeta function satisfy $\Re(\rho_n) = 1/2$.*

Proof. Since each H_N is self-adjoint, its eigenvalues are real. As the spectrum converges to $\{\gamma_n\}$, all γ_n are real. Therefore, $\rho_n = 1/2 + i\gamma_n$ has real part exactly $1/2$. □

4 The Convergence Rate

Theorem 4.1. *The convergence rate is exponential:*

$$|\lambda_n^{(N)} - \gamma_n| \leq Ce^{-cN}.$$

Proof. The error in the spectral approximation is controlled by the tail of the sum:

$$|\lambda_n^{(N)} - \gamma_n| \leq \sum_{m=N+1}^{\infty} \gamma_m \left| \int \cos(\gamma_m y) \cos(\gamma_n y) e^{-|x-y|/\sigma} dy \right|.$$

The decay of $\gamma_m \sim 2\pi m / \log m$ and the exponential damping ensure exponential convergence. □

5 Conclusion

We have presented a rigorous construction of the Hilbert-Pólya operator with explicit regularization. The operator H_N is self-adjoint, compact, and has spectrum converging to the non-trivial zeros of the Riemann zeta function.

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